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ITMP

On AdS3 Holography

Pure, twisted, stringy and black.

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Plan: A conceptual tour (2019-2022)

Part 1: * Pure AdS3 Gravity : A Proposal for the Dual

Part 2:

* Pure AdS3 Supergravity: Topological AdS/CFT in a Toy Model
* Topological AdS/CFT in String Theory
* String Theory in AdS3 x S1: further implementation of topological AdS/CFT

Part 3: * A Step Towards String Theory in BTZ Black Holes

Based on and related to work by: Brown, Carlip, Costello, Coussaert, Dei, Eberhardt, Eguchi, Gaberdiel, Giribet, Giveon, Gopakumar, Hemming, Henneaux, Keski-Vakkuri, Krasnov, Kraus, Kutasov, Li, Maldacena, Mertens, Ooguri, Pakman, Polyakov, Porrati, Rangamani, Rastelli, Ross, Seiberg, Strominger, Teitelboim,Tseytlin, Van Driel, Witten, …

and many other authors whom I gratefully acknowledge.

Pure AdS3 Gravity : A Proposal

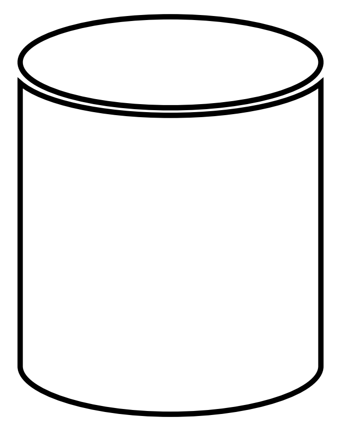
Three-dimensional gravity is topological.

The Einstein-Hilbert action can be written as a Chern-Simons theory.

The gauge fields Γ are the sums and differences of the spin connection ω and the dreibein e .

With negative cosmological constant, the gauge algebra is $so(2,2) = so(2,1) \oplus so(2,1)$.

The gauge fields satisfy Brown-Henneaux boundary conditions.



The boundary action consists of two chiral sl(2,R) WZWN actions.

The boundary conditions gauge a null direction, reducing the boundary dynamics to one scalar field with a (chiral) Liouville action. Glue the zero modes.

Bulk	Boundary

$$S_{\text{Einstein-Hilbert}} = S_{\text{Liouville}}$$

Proposal: define the dual theory of pure AdS3 gravity to be the two-dimensional Liouville conformal field theory.

Draw the consequences of the tightly constrained proposal.

Advantages:

Manifestly consistent and unitary CFT dual

Three-point functions (black hole scattering)

Crossing

…

Disadvantages:

No black hole entropy

An isolated theory
of low dimensional quantum gravity.

Origin of Liouville measure ?

There are (BTZ) ‘black hole’ solutions.

There is no thermodynamics/ statistical mechanics/ Hawking radiation.

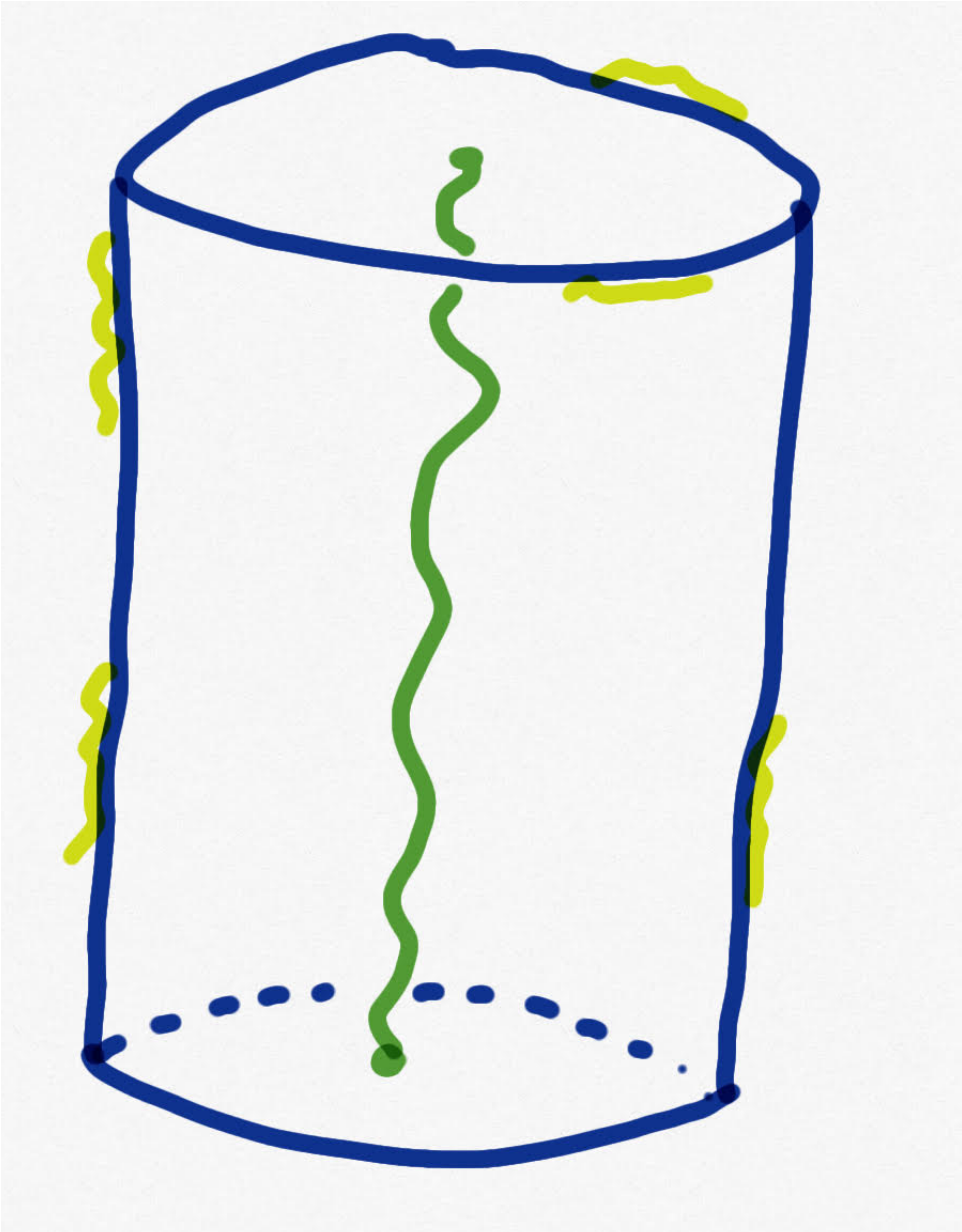
Consequences of the proposal:

The only primary states in the theory are primary black holes, i.e. with a mass above a mass gap (proportional to the central charge).

All other states are descendants of these states, made up of boundary gravitons.

There is no AdS3 ground state in the system. The path integral is not normalisable.

Takeaway 1



End of part I

Topological AdS/CFT
in Pure AdS3 Supergravity

General problem:

Quantum field theories with extended supersymmetry can be twisted:

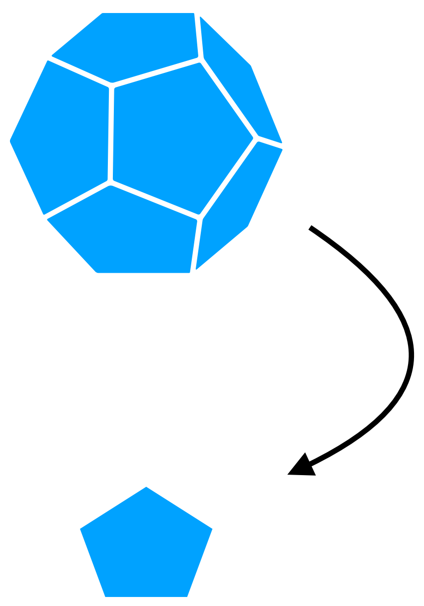
a subsector can be defined as a topological quantum field theory,
which allows to compute a very small subset of the observables of the original theory.

Suppose we twist a CFT dual to quantum gravity in AdS.

What is the bulk dual of the topological twist ?

How does one define a twisted theory of quantum gravity ?

Pure Supergravity in AdS3 <-> N=2 Liouville Theory



Twisting in N=2 Superconformal Field Theories in D=2

$$(T, G^{\pm}, J_R)$$

Spin/dimension (2,3/2,3/2,1)

$$G^{+}_{\frac{1}{2}} = \oint G^{+}$$ Dimension 1/2.

Twist: $$T_{\text{top}} = T - \frac{1}{2}\partial J_R$$

Scalar Supercharge $\qquad G^{+}_{\frac{1}{2}} \xrightarrow{\text{twist}} Q$

Defines cohomology and space of observables of topological theory.

How to twist in the bulk ?

The bulk supergravity theory will satisfy twisted boundary conditions.
The bulk metric couples to the twisted energy-momentum tensor provided we introduce metric dependent twists in the boundary conditions on the gauge fields in the three-dimensional supergravity theory.

osp(1,1|2)

Brown-Henneaux

Boundary gravitons

Boundary U(1) R gauge field

$$\Gamma \xrightarrow[r\to\infty]{} \left(\frac{4\pi L}{k_R}\frac{\sigma^+}{r} + r\sigma^- + \frac{1}{\sqrt{r}}\frac{4\pi Q_{+\alpha}}{k_R}R^{+\alpha} + \frac{2\pi B}{k_R}T\right)dx^+ + 0\,dx^- + \frac{\sigma^3}{2}\frac{dr}{r}$$

Boundary gravitinos

$$g_{(0)\mu\nu} = e^{2\omega}\eta_{\mu\nu}$$

Conformally flat AND Twisted

$$\Gamma \xrightarrow[r\to\infty]{} \left(-\partial_+\omega\frac{\sigma^3}{2} + \frac{4\pi L}{k_R}\frac{\sigma^+}{r} + re^{\omega}\sigma^- + \frac{1}{\sqrt{r}}\frac{4\pi Q_{+\alpha}}{k_R}R^{+\alpha} + \frac{2\pi B}{k_R}T\right)dx^+$$
$$+\left[\partial_-\omega\left(\frac{\sigma^3}{2} + \frac{iT}{2}\right) + \frac{1}{re^{\omega}}\partial_+\partial_-\omega\sigma^+\right]dx^- + \frac{\sigma^3}{2}\frac{dr}{r}$$

The Twist

A calculation gives the boundary action

Topologically twisted N=2 Liouville theory

Boundary gravitons

Boundary U(1) R gauge field

$$S^{top}_{Liouville} = \frac{k_R}{4\pi}\int dx^+dx^-\Big(\partial_+\rho\partial_-\rho + \partial_+\theta\partial_-\theta + ie^{\omega}\bar{\psi}_+\partial_-\psi_+ + ie^{\omega}\psi_-\partial_+\bar{\psi}_-$$
$$+e^{2\omega}[e^{2\rho} - i(e^{\rho+i\theta}\bar{\psi}_+\psi_- + e^{\rho-i\theta}\psi_+\bar{\psi}_-) + \frac{1}{4}(\rho+i\theta)\mathcal{R}^{(2)}]\Big)\,.$$

Boundary gravitinos

Topological

Localize Gravitational Path Integral

On

(Known) Gravitational Chiral Primaries

⚠️🚧

Topological AdS/CFT in String Theory

Extra challenge: twist the boundary conditions on-shell in the bulk. Therefore, solve the supergravity equations with general boundary metric and flat bulk U(1) R-symmetry gauge field.

Then set the U(1) R gauge field equal to the spin connection.

Consequence: the world sheet operator that represents the space-time energy-momentum tensor is of the twisted form.

Compute the cohomology. (Largely done.)
Localize the string path integral. (Largely undone.)

In more detail: on-shell string theory

Fefferman-Graham + NSNS Flux

$$
\begin{aligned}
ds^2 &= l^2\left(\frac{dr^2}{r^2} + (r^2 g^{(0)}_{ij} + g^{(2)}_{ij} + r^{-2} g^{(4)}_{ij}) dx^i dx^j\right) \\
H_{(3)} &= \frac{2}{l}\sqrt{|G|}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \\
\Phi &= \text{constant}\,.
\end{aligned}
$$

On-shell

Add a circle. AdS3 x S1. KK reduce to find U(1) R Gauge Field.

Global conformal and U(1)_R boundary symmetry in bulk geometric terms.

$$\begin{aligned} \Phi &: \text{constant} \\ G_{\mu\nu} &: \text{locally } AdS_3 \text{ with a non-trivial boundary metric} \\ H &: \text{proportional to the volume form} \\ G_{44} &: \text{constant} \\ A^R \text{ and } \bar{A}^R &: \text{flat}\,. \end{aligned}$$

$$A^R_{\bar{x}} = -\frac{i}{4}\omega^{+-}_{\bar{x}}\,, \qquad \bar{A}^R_x = \mp\frac{i}{4}\omega^{+-}_x$$

R-symmetry gauge field

The Twist

Spin connection

We demonstrated that

the (generic, on-shell) variation of the world sheet partition function with respect to the boundary metric (around a conformally flat background)

leads to

the insertion of the topologically twisted space-time energy-momentum tensor (world sheet vertex operator).

AdS3 x S1 String Theory

The 1/2 BPS cohomology in all sectors was classified.

(Preparation for localisation of path integral: open problem.)

We computed space-time chiral ring structure constants in the zero winding sector using world sheet operator product expansion, finding surprising properties, reminiscent of (for now non-existing) N=2 D=2 non-renormalisation theorems.

(Preparation for having a purely topological calculation.)

Takeaway 2

First Step: Twist the boundary conditions on the R-symmetry gauge field with the spin connection to topologically twist quantum gravity in asymptotically anti-de Sitter spacetimes.

Second Step: Compute the bulk scalar cohomology and localise the path integral. (WIP)

End of part II

Part III:

A Step Towards Understanding String Theory in BTZ Black Holes.

$$ds^2 = \frac{l^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 - \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2} dt^2 + r^2 (d\phi - \frac{r_+ r_-}{l r^2} dt)^2$$

BTZ black holes are Z orbifolds of the universal covering group of SL(2,R)

$$x_{-1}^2 + x_0^2 - x_1^2 - x_2^2 = 1$$

$$g \to e^{\pi (r_+ - r_-)\sigma_3} \, g \, e^{\pi (r_+ + r_-)\sigma_3}$$

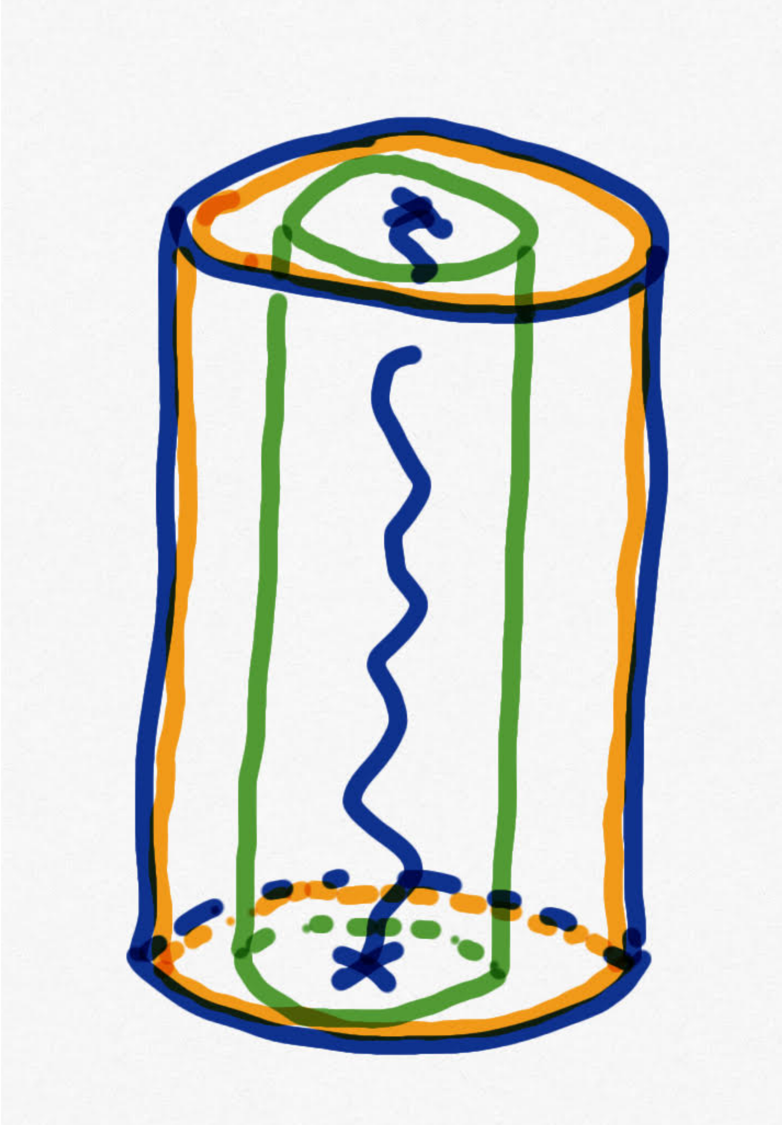
(NSNS) String Theory on BTZ black holes may be described by an orbifolded world sheet WZWN model on the group manifold.

WZWN term

$$H_{(3)} = r dr \wedge dt \wedge d\phi \qquad\qquad B_{(2)} = \frac{1}{2}(r^2 - c) dt \wedge d\phi$$

The WZWN term is hard to define unambiguously on the plane times circle topology of the black hole.

The integral of the three-form requires a three-manifold.

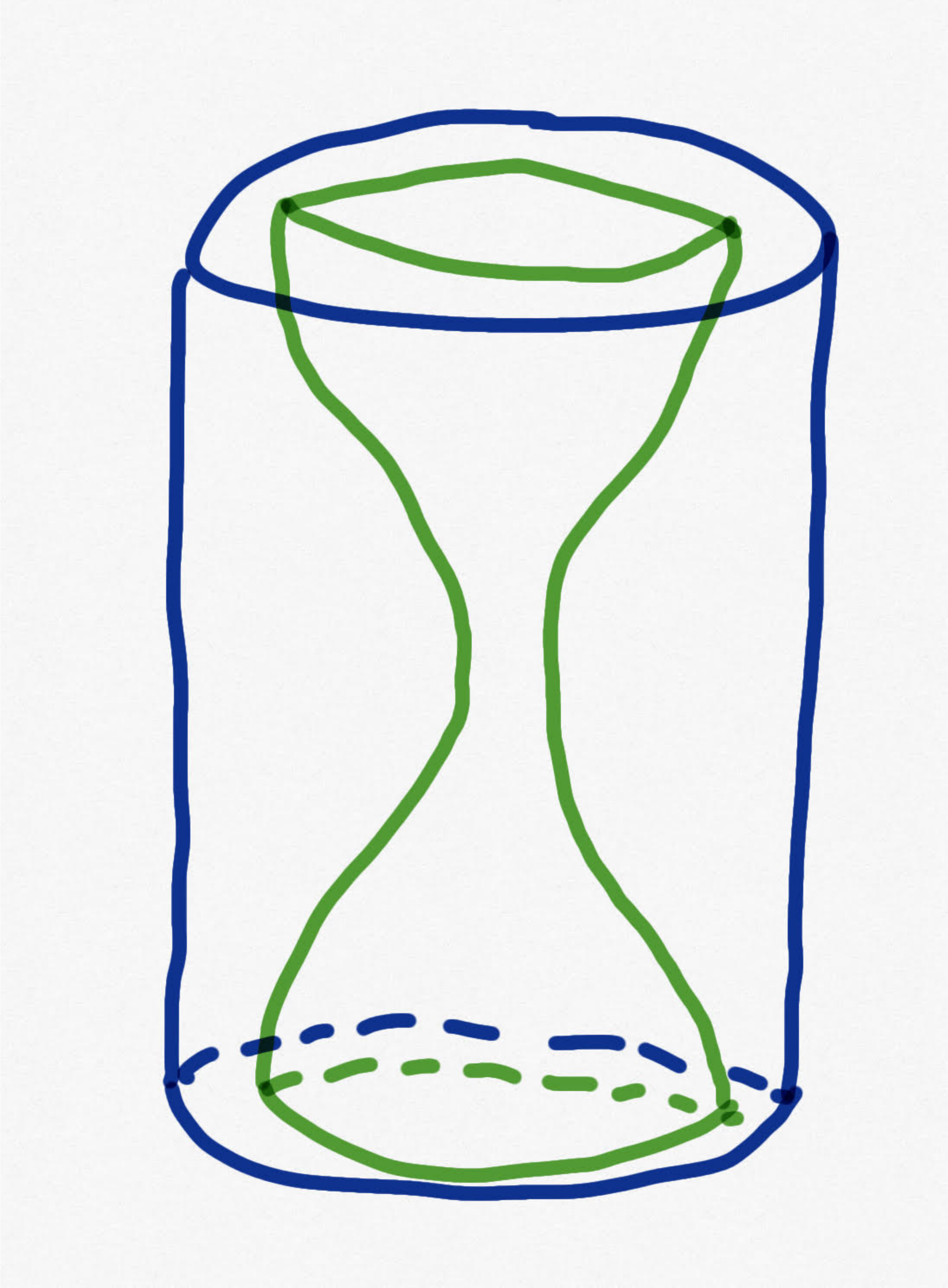


$$\begin{aligned}\int_{\text{cylinder}} H_{(3)} &= \int_{\Sigma} B_{(2)} - \int_{\Sigma_{co}} B_{(2)} \\ &= \frac{1}{2}\int_{\Sigma}(r^2 - c) dt d\phi - \frac{1}{2}\int_{\Sigma_{co}}(r_{co}^2 - c) dt d\phi \\ &= \frac{1}{2}\int (r^2 - r_{co}^2) dt d\phi\,.\end{aligned}$$

The integral of the two-form contains an integration constant.

We study Winding Strings
(Part of the asymptotic states)

$$t(\tau,\sigma) = t_{\text{geodesic}}(\tau) + w\tau$$
$$\phi(\tau,\sigma) = \phi_{\text{geodesic}}(\tau) + w\sigma$$
$$r(\tau) = r_{\text{geodesic}}(\tau)$$



For large winding strings, the infinite volume contribution is cancelled by an infinite flux contribution. We are (still) left with a finite ambiguity in the energy.

We compute the energy of an asymptotic winding string in two ways:

Using an (ambiguous) Nambu-Goto + flux action (in static gauge)

And the standard WZWN construction + Z-orbifold (in conformal gauge).

$$S = S_{NG} + S_B$$

Nambu-Goto action

$$\tau = t \ , \qquad \phi = w\sigma$$

Static Gauge

Large radius expansion of the Action

$$S = \frac{kw}{4\pi} \int d\tau d\sigma \left(r_+^2 + r_-^2 - 2c) + \frac{1}{r^2} \left(\dot{r}^2 - \frac{1}{w^2}(r')^2 + \frac{1}{4}(r_+^2 - r_-^2)^2 \right) \right) + \dots$$

$$\dot{r} = 0 = r' \qquad r = r_{\max} \qquad r_- = 0$$

Compute energy when maximal radius is reached and the black hole does not rotate.

Final result of NG approach:

$$E_{\text{string}} = -\int d\sigma \; L = -kw \left((r_{\max}^2 - c) - r_{\max}\sqrt{r_{\max}^2 - r_+^2} \right)$$

Second approach: WZWN (conformal gauge) + solve Virasoro constraints

(Retain integrability, holomorphy, affine symmetry, ..)

$$E_{\text{string}} = -k\, w\, r_{\max} \left(r_{\max} - \sqrt{r_{\max}^2 - r_+^2} \right)$$

Algebraic structure of the world sheet model resolves the ambiguity:

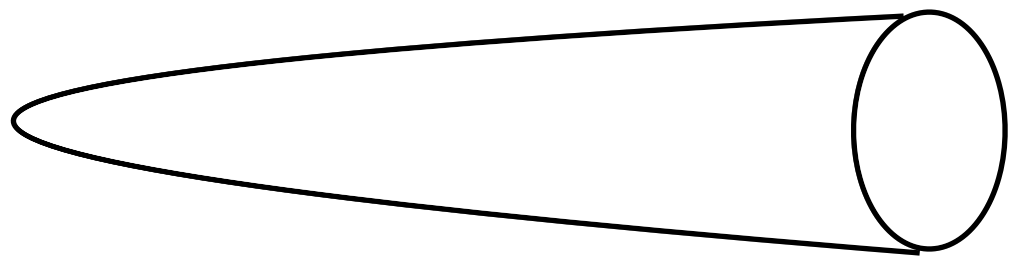
$$c = 0$$

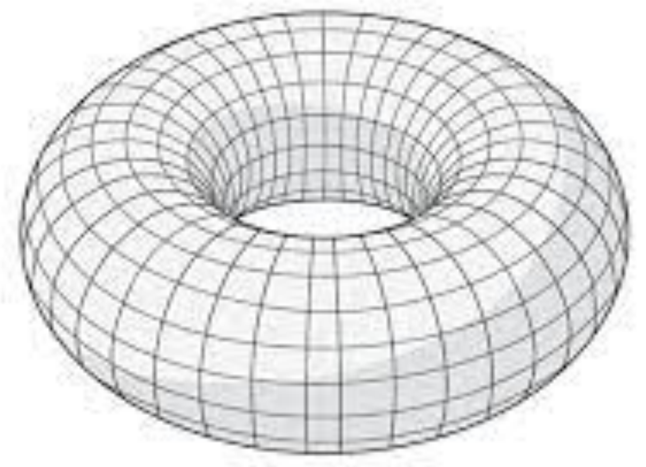
We also explored the partition function of the world sheet model.

The partition function is computed in the Euclidean.

Regularity at the Euclidean point-like horizon requires

$$c = r_+^2$$





A Modular S-transformation on the Torus Boundary relates the thermal AdS3 and the euclidean BTZ geometries

Distinct Dual Interpretations

$$Z_{EAdS_3}(\beta, \mu\beta) = Tr_{\mathcal{H}} e^{-\beta H + i\mu\beta L}$$

Weighted sum of states

Thermal Circle

(Trivial angle)

$$Z_{BTZ(M,J)} = \sum_{w} Tr_{\mathcal{H}_w} e^{2\pi i w L}$$

Orbifold projection

Angular Identification

(Trivally thermal)

The result for the world sheet partition function on the BTZ orbifold geometry is:

$$Z_{BTZ} = \frac{r_+\sqrt{k_b - 2}}{\sqrt{\tau_2}} \sum_{w,m} \frac{e^{-\pi \frac{k_b}{\tau_2} r_+^2 |m - w\tau|^2 + \frac{2\pi}{\tau_2} \mathrm{Im}(\bar{U}_{m,w})^2}}{|\theta_1(\bar{U}_{m,w}, \tau)|^2} \qquad \bar{U}_{m,w} = (r_- - ir_+)(m - w\tau)$$

(For thermal AdS3, interpretable as the thermal partition function corresponding to the Lorentzian spectrum.)

Naive expansion gives, after analytic continuation to the Lorentzian, discrete representations which match short strings, appropriately projected.

Moreover, we interpreted a remainder

$$Z_{BTZ}^{div} = \frac{r_+\sqrt{k_b-2}}{\sqrt{\tau_2}} \sum_{w,m} \frac{1}{|\eta(\tau)|^6} \delta(\bar{U}_{m,w})\delta(U_{m,w})$$

modular invariant pole contribution that can be identified
with an integral over twisted sector continuous characters.

More work is needed to attain the degree of semi-rigour achieved
in the analysis of the (thermal and Lorentzian) AdS3 string theory spectrum.

End of part 3

Tour Terminus

Pure gravity	Consistent but isolated !?
Topological AdS/CFT	R-symmetry Gauge Field set by Metric
Strings in Black Holes	Insight into Asymptotic Winding States

Research domains in which further substantial progress can be made with elementary means.